

# QUALITATIVE ANALYSIS OF A SCALAR-TENSOR THEORY WITH EXPONENTIAL POTENTIAL

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## Abstract

A qualitative analysis of a scalar-tensor cosmological model, with an exponential potential for the scalar field, is performed. The phase diagram for the flat case is constructed. It is shown that solutions with an initial and final inflationary behaviour appear. The conditions for which the scenario favored by supernova type Ia observations becomes an attractor in the space of the solutions are established.

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## 1 Introduction

The results from the high redshift supernova type Ia observations [1, 2] indicate that the Universe is in an accelerated expansion regime. This means that the matter content of the Universe must be dominated by an exotic fluid whose pressure is negative with  $p < -\frac{2}{3}$ . Generally, the cosmological constant, with  $p = -\rho$ , is assumed as the most natural candidate to represent this exotic fluid. However, the theoretical problems concerning a cosmological constant, somehow connected with its interpretation as a vacuum energy [3], make other possibilities very attractive also.

Some kinds of topological defects can also lead to an effective equation of state in the searched range. Non-relativistic domain walls, for example, implies  $p = -\frac{2}{3}\rho$ . But, a more fashion theoretical proposal is the so-called quintessence model [4], which is a minimal coupled scalar field with a slow evolving potential. It has been argued that such kind of potentials may originate from supergravity models, and the accelerated solutions corresponds to an attractor [5]. These kind of models may be tracked back to the reference [6], where they have been proposed in order to solve the dark matter puzzle. Quintessence model makes use of a large variety of forms for the potential: exponential[4], polynomial combined with exponential [5], hyperbolic sine or cosine [7] or even a double exponential [8].

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In a recent work [9], it has been showed that an exponential potential model admits as particular solution the typical perfect fluid solution of the Friedmann-Robertson-Walker model, for any value of the barotropic equation of state parameter  $\alpha$  ( $p = \alpha\rho$ ). Hence, inflationary power-law solutions are covered by this model. One of our interest here is to study the status of this particular solutions, verifying to which extent they are attractors, and if it is possible to obtain models where the expansion is initially non-accelerated, becoming accelerated later. Due to the fact that the standard cosmological model needs an inflationary phase in its primordial phase, it would be also interesting to have models where an initial and final inflationary phase occur, with an intermediate non-inflationary behaviour, in order not to spoil the nucleosynthesis achievements and structure formation.

In order to do so, we will perform a qualitative analysis of this model, first in the scalar-tensor model, and secondly with the scalar-tensor model coupled to ordinary matter. Some studies have already been made for these cases in the literature [10, 11, 12]. However, in [10] it has been mainly verified that the flat case is an attractor for a great variety of exponential factor, while in [11, 12] the analysis, with respect to this exponential model, was mainly dedicated to the identification of the nature of critical points.

In the present case, we will be interested in mapping completely the solutions for  $k = 0$ , since it has already been shown that  $k = 0$  is an attractor with respect to  $k = \pm 1$ . Another reason is that  $k = 0$  seems to be favoured by observations [13]. The phase diagram will be constructed and the positive energy and inflationary regions will be identified. It comes out that the both kind of desirable models, as described before, appear and the power-law particular solutions are indeed attractors of the physical acceptable solutions.

In next section we describe with some detail the scalar-tensor model with exponential potential. In section 3, the phase diagram is constructed. The coupling to ordinary matter is discussed in section 4, while in section 5 the conclusions are presented.

## 2 The scalar-tensor model

A minimal coupling between gravity and a self-interacting scalar field is represented by the lagrangian,

$$L = \sqrt{-g} \left[ R - \phi_{;\rho} \phi^{;\rho} + 2V(\phi) \right] \quad . \quad (1)$$

The field equations are

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \phi_{;\mu} \phi_{;\nu} - \frac{1}{2} g_{\mu\nu} \phi_{;\rho} \phi^{;\rho} + g_{\mu\nu} V(\phi) \quad , \quad (2)$$

$$\square \phi = -V'(\phi) \quad , \quad (3)$$

where the prime means derivative with respect to  $\phi$ . Inserting the FRW metric

$$ds^2 = dt^2 - a^2(t) \left[ \frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \right] \quad (4)$$

where  $k = 0, -1, 1$  corresponds to a flat, open and closed Universe respectively, it results the following equations of motion:

$$3\left(\frac{\dot{a}}{a}\right)^2 + 3\frac{k}{a^2} = \frac{1}{2}\dot{\phi}^2 + V(\phi) \quad , \quad (5)$$

$$-2\frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2 - \frac{k}{a^2} = \frac{1}{2}\dot{\phi}^2 - V(\phi) \quad , \quad (6)$$

$$\ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} = -V'(\phi) \quad . \quad (7)$$

These equations are connected by the Bianchi identities. If the potential term is written as

$$V(\phi) = \frac{2}{3} \frac{1 - \alpha}{(1 + \alpha)^2} \exp(\mp \sqrt{3(1 + \alpha)}\phi) \quad , \quad (8)$$

where  $-1 \leq \alpha \leq 1$ , the equations of motion (5,6) admit, for the flat case, the solutions

$$a(t) = a_0 t^{\frac{2}{3(1+\alpha)}} \quad , \quad \phi(t) = \pm \frac{2}{\sqrt{3(1+\alpha)}} \ln t \quad . \quad (9)$$

Hence, the potential (8) leads to the usual perfect fluid solutions of Einstein equation as a particular case. When  $\alpha > -\frac{1}{3}$ , these particular solutions represent an expanding, decelerating Universe; if  $\alpha < -\frac{1}{3}$ , the scale factor describes an expanding, accelerating (inflationary) Universe. The limiting case  $\alpha = -\frac{1}{3}$  corresponds to  $a \propto t$ , that is,  $\dot{a} > 0$  and  $\ddot{a} = 0$ . So, the perfect fluid solutions may be mimitized by a scalar-tensor model with a suitable potential. In [9] it was shown that this scalar-tensor model allows to get rid of instabilities that appears in the perfect fluid models, at perturbative level and in the small wavelength limit, when  $\alpha < -\frac{1}{3}$ .

However, it is important to know if these particular solutions represent an attractor of the space of all possible solutions of this model. Moreover, the solutions (9) describe an Universe that is always inflationary ( $\alpha < -1/3$ ) or always non-inflationary ( $\alpha > -1/3$ ). In a realistic Universe, the behaviour of the scale factor must change with time. If we accept the inflationary paradigm for the early Universe and if the results for the value of the decelerating parameter  $q = -\frac{\ddot{a}a}{\dot{a}^2}$  obtained through the measurement of the supernova type Ia are confirmed, then the most realistic model should have an initial and final inflationary behaviour; between these two inflationary stages, a non-inflationary behaviour must take place, allowing the formation of light elements, through the primordial nucleosynthesis, and formation of local structure through the gravitational instability mechanism.

The model described by (1), with the potential given by (8), is rich enough to allow solutions more general than (9). In order to exploit all richness of this model, verifying if realistic models as described before are possible, we will perform a qualitative analysis of (5,6). Compactifying the phase space of all possible solutions on the Poincaré's sphere, and delimiting the regions corresponding to the positivity of the potential  $V(\phi)$ , as well as the inflationary type solutions, we will identify the main features of the cosmological models resulting from (5,6,7).

### 3 The phase space diagram

In order to perform a qualitative analysis of the system described above, we define the new variables

$$x = \dot{\phi} \quad , \quad y = \frac{\dot{a}}{a} \quad , \quad (10)$$

and we set  $B = \pm\sqrt{3(1+\alpha)}$  and  $V(\phi) = V_0 e^{B\phi}$ . Hence, equations (5,6,7) lead to the planar, homogenous, autonomous two dimensional system

$$\dot{x} = \frac{B}{2}x^2 - 3xy - 3By^2 \quad , \quad (11)$$

$$\dot{y} = -\frac{1}{2}x^2 \quad , \quad (12)$$

subjected to the condition

$$V(\phi) = y^2 - \frac{1}{6}x^2 > 0 \quad . \quad (13)$$

This system admits in the finite region of the plane  $(x, y)$  an unique degenerate critical point  $x = 0, y = 0$ . It corresponds to the Minkowski space. Its eigenvalues are all zero.

Now, the invariant rays in this plane are characterized by the solutions  $x = \lambda y$ . Inserting this relation in (11,12) it results a third order polynomial relation for  $\lambda$ :

$$\lambda^3 + B\lambda^2 - 6\lambda - 6B = 0 \iff (\lambda + B)(\lambda^2 - 6) = 0 \iff \lambda_1 = -B \quad , \quad \lambda_{\pm} = \pm\sqrt{6} \quad . \quad (14)$$

Hence, there are three invariant rays for  $B \neq \pm\sqrt{6}$ . Two of them are independent of the value of  $B$  while the third one depends on  $B$  and, consequently, on  $\alpha$ . This third invariant ray correspond to the solution (9). Let  $[XY]$ ,  $[CC']$  and  $[AA']$  denote respectively the rays  $x = -By$ ,  $x = -\sqrt{6}y$  and  $x = \sqrt{6}y$ .

Inserting the expressions for these invariant rays into (12) the direction of evolution along them can be determined. In order to complete the analysis, the critical point at infinity must be found and their nature studied [14]. To do this, we project the plan  $(x, y)$  into the Poincaré's sphere, introducing new variables  $z, u, v$ , such that  $x = \frac{u}{z}$ ,  $y = \frac{v}{z}$  subjected to the condition  $u^2 + v^2 + z^2 = 1$ . The new system reads:

$$\frac{du}{d\tau} = bz - cv \quad , \quad (15)$$

$$\frac{dv}{d\tau} = cu - az \quad , \quad (16)$$

$$\frac{dz}{d\tau} = av - bu \quad , \quad (17)$$

where  $a = \frac{1}{2}u^2z$ ,  $b = z(\frac{B}{2}u^2 - 3uv - 3Bv^2)$  and  $c = -\frac{1}{2}u^3 - \frac{B}{2}u^2v + 3uv^2 + 3Bv^3$ . The new parameter  $\tau$  is defined by the equation (17). The points at infinity are obtained for  $z = 0$  which corresponds to the equator of the Poincaré's sphere. The critical points are

determined and their nature (repulsive, attractive or saddle points) is characterized in the usual way [14]. The coordinates of these points are

$$\begin{aligned} A(u, v, z) &= \left( \sqrt{\frac{6}{7}}, \frac{\sqrt{7}}{7}, 0 \right) \quad , \quad A'(u, v, z) = \left( -\sqrt{\frac{6}{7}}, -\frac{\sqrt{7}}{7}, 0 \right) \quad , \\ C(u, v, z) &= \left( \sqrt{\frac{6}{7}}, -\frac{\sqrt{7}}{7}, 0 \right) \quad , \quad C'(u, v, z) = \left( -\sqrt{\frac{6}{7}}, \frac{\sqrt{7}}{7}, 0 \right) \quad , \\ X(u, v, z) &= \left( \frac{-B}{\sqrt{1+B^2}}, \frac{1}{\sqrt{1+B^2}}, 0 \right) \quad , \quad Y(u, v, z) = \left( \frac{B}{\sqrt{1+B^2}}, \frac{-1}{\sqrt{1+B^2}}, 0 \right) \quad . \end{aligned}$$

The final phase diagram contain six regions separated by the three invariant rays. Three of the six regions may be obtained from the other three by inverting the time:  $t \rightarrow -t$ . In order to interpret the behaviour and the nature of the curves, we complement this diagram with some other physical considerations.

The complete diagram depends on the value of  $B$ . However, before establish these diagrams two physical requirements will be introduced. The first one concerns the positivity of the contribution of the energy of the potential term. The potential  $V(\phi)$  is negative in the region interior to the rays such that  $\lambda = \pm\sqrt{6}$ . In this region the variable  $x$  can never be zero, since in this case the variable  $y$  becomes imaginary.

We will be also interested in identifying the regions where inflation can occurs. This implies to require  $\ddot{a} > 0$ . Hence,  $\dot{y} + y^2 > 0$ . Due to (12), this implies  $-\frac{1}{2}x^2 + y^2 > 0$ . Hence, the inflationary region is bounded by the rays  $y = \pm\frac{\sqrt{2}}{2}x$ . These rays will be represented by [PP'] and [QQ']. The inflationary regime occurs in the region bounded by these lines and containing the  $y$  axis.

Hence, in these phase diagram the two invariant rays (from three), the condition of positivity of the energy, and the condition to have inflation are fixed and independent of  $B$ . Only one invariant ray depends on  $B$ ; it corresponds to the particular solution (9). When  $\alpha = 1$ , this invariant ray coincides with one of other two.

The complete diagram is displayed in figure 1 for  $-\sqrt{2} < B < 0$ . As  $B$  changes from  $-\sqrt{6}$  to  $\sqrt{6}$ , the invariant ray [XY] moves from [CC'] to [AA']. When  $\alpha = -1$ , the invariant ray [XY] coincides with the  $y$  axis.

If  $-\frac{1}{3} < \alpha < 1$ , ( $-\sqrt{6} < B < -\sqrt{2}$  or  $\sqrt{2} < B < \sqrt{6}$ ), the particular perfect fluid-type solutions are non-inflationary. There are two physically interesting kind of trajectories in this case. Those connecting the critical point at infinity  $C$  to the origin, approaching asymptotically the invariant ray [YX], start with a free scalar field behaviour ( $a \propto t^{1/3}$ ), and coincide later with the corresponding perfect fluid solution; these curves describe non-inflationary Universe. However, the curves connecting the critical point at infinity  $Y$  to the origin, start with a non-inflationary perfect fluid behaviour, becomes latter inflationary and assymptotically coincides again with the corresponding non-inflationary perfect fluid solution.

When  $-1 < \alpha < -\frac{1}{3}$ , which corresponds to  $-\sqrt{2} < B < \sqrt{2}$ , the perfect fluid-type particular solutions corresponding to the invariant ray [XY] are inflationary. Again, we have two kind of interesting trajectories. The solutions starting from the critical point

at infinity  $C$  begins again with a free scalar field, non-inflationary, behaviour and then evolves towards an inflationary regime as they approach the origin. Those solutions starting from the critical point  $Y$ , have initially an inflationary behaviour, becomes later non-inflationary and again become inflationary as they approach the origin along the invariant ray  $[XY]$ . There are also solutions that begin at  $Y$ , ending in the origin, remaining always inflationary.

These last trajectories are the most interesting one since the standard cosmological model requires an initial inflationary regime, as well as a final inflationary regime, if the supernova type Ia measurements are confirmed.

When  $B = 0$  ( $\alpha = -1$ ) all  $y$  axis becomes singular and corresponds to the de Sitter particular solution with different values for the Hubble factor  $\frac{\dot{a}}{a}$ . All physical acceptable solutions go to or come from one point of this singular axis. It is important to notice that, for any value of  $B$ , the invariant ray  $[XY]$  is an attractor.

## 4 Coupling the scalar field to ordinary matter

The inclusion of ordinary perfect fluid matter, with a barotropic equation, can be made in a quite direct way. In this case, the equations of motion read

$$\left(\frac{\dot{a}}{a}\right)^2 = 8\pi G\rho + \frac{1}{2}\dot{\phi}^2 + V(\phi) \quad , \quad (18)$$

$$-2\frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2 = 8\pi G\gamma\rho + \frac{1}{2}\dot{\phi}^2 - V(\phi) \quad , \quad (19)$$

$$\ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} = -V_\phi(\phi) \quad , \quad (20)$$

$$\dot{\rho} + 3(1 + \gamma)\frac{\dot{a}}{a}\rho = 0 \quad . \quad (21)$$

In these expressions, the barotropic equation of state  $p = \gamma\rho$  was explicitly employed.

The equations of motion above may be recast in the form of a three-dimensional dynamic system:

$$\dot{x} = Bz + \frac{B}{2}x^2 - 3xy - 3By^2 \quad , \quad (22)$$

$$\dot{y} = -\frac{1+\gamma}{2}z + x^2 \quad , \quad (23)$$

$$\dot{z} = -3(1 + \gamma)yz \quad . \quad (24)$$

The definition of  $x$  and  $y$  are the same as before and  $z = 8\pi G\rho$ . For  $z = 0$ , we go back to the two-dimensional system (11,12).

The study of a three-dimensional dynamical system is more involved. However, we may obtain its general features following closely the procedure employed in the two-dimensional case. First of all, we remark that the system (22,23,24) has one critical

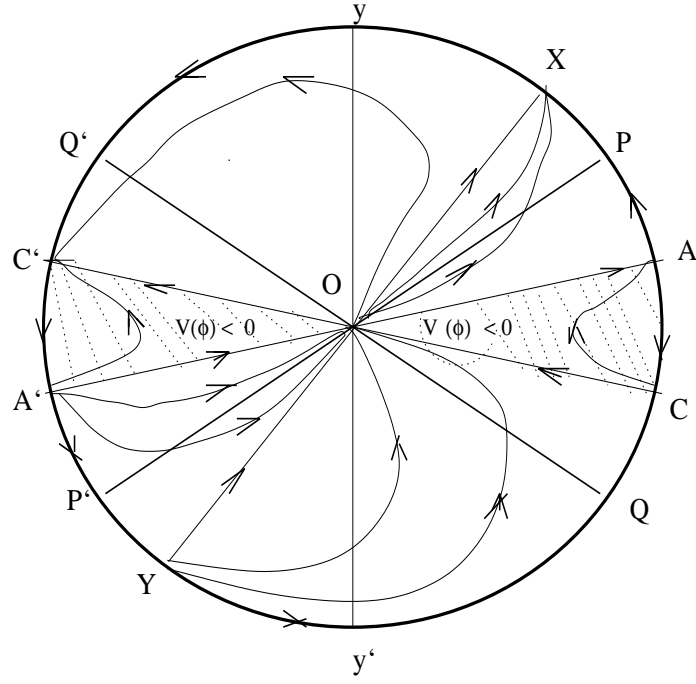


Figure 1: *This diagram gives the Evolution of different solutions through inflationary or non inflationary regime. It shows the reheating phenomena for some solutions in the sector OX and YO*

point, as before, represented by  $x = 0$ ,  $y = 0$  and  $z = 0$ . It corresponds to the Minkowski space. But, in order to obtain the invariant rays and the critical points at infinity, we must perform some suitable "cuts" in the three-dimensional phase diagram, reducing it to an ensemble of two-dimensional system which can then be completely analyzed. These "cuts" correspond to projections of the three-dimensional trajectories on some planes.

The most simple and natural projections correspond to impose  $x = 0$ ,  $y = 0$  or  $z = 0$ , respectively. The last one leads to the two-dimensional system analyzed in the previous section. On the other hand, when we impose  $x = 0$  (no scalar field) or  $y = 0$  (no gravity), the resulting system can be completely solved. Specifically, these hypothesis lead to

$$x = 0 \rightarrow y = \sqrt{\frac{z}{3}} \rightarrow a \propto t^{\frac{2}{3(1+\alpha)}} \quad , \quad (25)$$

$$y = 0 \rightarrow x = z = 0 \quad (\alpha \geq -1) \quad . \quad (26)$$

The first case corresponds to the gravity perfect fluid system, whose solution is well known. The second one corresponds to the trivial case: the Minkowski solution can exist only if matter and the scalar field are absent also.

The fact that those two cases are completely solved (one of them through the trivial solution) just mean that if the solution is initially in a plane  $x = 0$  or  $y = 0$ , it remains there. Hence, the most interesting solutions, with non trivial solutions for  $x$ ,  $y$  and  $z$ , are those outside these planes: however, their projection on the  $z = 0$  behaves as described in the previous section, and all analysis performed before remains.

## 5 Conclusions

A self-interacting scalar field coupled to gravity is considered as a good candidate to describe the dominant matter content of the Universe today, leading to an accelerate expansion. The potential term in general is taken as an exponential function of the scalar field, or a combination of polynomial and exponential functions. Here we have exploited a model where the potential is just an exponential function of the scalar field. In [9] it was shown that such potential term may lead to power-law solutions for the scale factor typical of a perfect fluid gravity system, with an arbitrary barotropic equation of state. Hence, inflationary power-law solutions are included in this scalar-tensor model.

In the present work, we performed a dynamical analysis of that model for a flat Universe. It was verified that the power-law particular solutions act as attractor in the space of the allowable solutions. Moreover, it was identified, for some exponential factors, solutions with an initial and final inflationary behaviour as well as solutions with just a final inflationary behaviour. They coincide asymptotically with the particular power-law solutions.

The restriction to  $k = 0$  was made due to two reasons mainly: in [10], it was shown that in an exponential potential scalar-tensor model, the flat case is in general an attractor. Moreover, the CMB anisotropy observations favor a flat Universe [13]. The goal of the present work was to present a complete phase diagram description for this particular, but very important case.



The results showed the richness of the model, and that such a simple self-interacting scalar field may lead to scenarios that are consistent theoretically and can be in good agreement with observations. In particular, if we admit a potential of the type  $V(\phi) = V_0 e^{\pm B\phi}$ , with  $B = 1$ , the scale factor behaves asymptotically as  $a \propto t^2$ , which is one of the most likely behaviours for the scale factor as can be inferred from the supernova type Ia observational programs [15].

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